Finite-State Markov Modeling of Diversity Nakagami Channels

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Abstract—**We design a Nakagami multi-channel simulator by modeling the received combined signal-to-noise ratio as a finite-state Markov chain, following a previously proposed approach. Our model generates directly the error process at the output of a diversity receiver and can emulate selection, maximal-ratio and equal-gain combining. As the order of diversity increases, the savings in computational complexity augment linearly with respect to a traditional waveform simulator.**

I. Introduction

The assumption that a fading channel can be modeled as a Markov chain can be used to design various low-complexity simulators. Instead of generating the correlated envelope of the channel, which then needs to be processed to determine if a decision error is made, one can directly produce the error process seen at the output of the demodulator. For over half a century, researchers have tried to fit such discrete-time models to realistic radio channels. These models range from the early Gilbert-Elliott two-state channels [1] to more complex models such as those based on hidden Markov chains [2]. The Finite State Markov Channel (FSMC) proposed in [3] has attracted quite some attention due to its good balance between accuracy and complexity. It is based on the partitioning of the received signal-to-noise ratio (SNR) in a finite number of states. The use of a first-order Markov process to model the envelope of a Rayleigh channel has been shown to be a good approximation in [4], using an information theoretic criterion, and was discussed recently in [5]. A higher-order model was proposed to represent Nakagami channels in [6], however the complexity of the calculations (requiring the numerical evaluation of double integrals) make its use less attractive. In previous papers FSMC's were mostly designed to model flat fading channels. If one needs to simulate the multiple paths of a channel in order to take into account diversity, the execution time of a waveform simulator increases with the number of diversity branches. Instead, one can generate directly the error process seen at the output of the diversity receiver. This technique was used in [7] for selection combining (SC) and Rayleigh fading, and was slightly discussed in [6] for an approximation to equal-gain combining (EGC) in Nakagami fading. Our works tackles the design of FSMC's for selection, maximal-ratio and equal-gain combining, in a generalized fading (Nakagami) environment. In the next section we derive the analytical steady state, transition and error probabilities for these three combining methods, in addition to the non-diversity case. These are integrated in a low-complexity simulator, whose first and second order statistics are compared with theoretical expressions in Section III.

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II. FSMC's for diversity Nakagami channels

At first we review the steps needed to design a FSMC, for arbitrary specifications of the channel and combining method. In later sections we specialize the model to a Nakagami fading channel without, and with SC, MRC and EGC diversity.

A. Review of FSMC Model

We use the approach first proposed in [3] to construct a FSMC. Let r be the *combined* envelope of the channel at the output of the diversity receiver, and $\gamma = r^2 E_b/N_0$ the postdetection SNR per symbol of the received signal. Let $p_\Gamma(\gamma)$ and $F_\Gamma(\gamma) = \int_0^\gamma p_\Gamma(\alpha)d\alpha$ be the probability density (pdf) and cumulative density (cdf) functions of γ. We define K partitions for γ such that if $\Gamma_k < \gamma < \Gamma_{k+1}$, $k = 0, \dots, K-1$, then the FSMC is said to be in the state s_k. The Γ_k's are the thresholds of the partition, with $\Gamma_0 = 0$ and $\Gamma_K = \infty$. A simple way of choosing these thresholds consists in specifying that the steady-state probabilities π_k of each state be all equal, i.e:

$$\pi_k = \int_{\Gamma_k}^{\Gamma_{k+1}} p_\Gamma(\alpha)d\alpha = F_\Gamma(\Gamma_{k+1}) - F_\Gamma(\Gamma_k) = \frac{1}{K} \quad (1)$$

for $k = 0, \dots, K-1$. The set of equations (1) must be solved numerically (or analytically if a closed-form solution exists) for the thresholds Γ_k, $k = 1, \dots, K-1$. This equal probability method (EPM) was proposed in [3]. Optimization of the thresholds using least squares quantization and the Lloyd-Max algorithm was later suggested in [8]. However, the latter requires much more computations, and as the number of states increases the advantage in accuracy with respect to the EPM diminishes. We thus rely on the EPM throughout this work. Let $\overline{\gamma} = E[\gamma] = \Omega E_b/N_0$ be the average SNR of the received signal, with $\Omega = E[r^2]$. The average SNR corresponding to the state k is then:

$$\bar{\gamma_k} = \frac{1}{\pi_k}\int_{\Gamma_k}^{\Gamma_{k+1}} \alpha p_\Gamma(\alpha)d\alpha \quad (2)$$

In a first-order Markov model, transitions are possible only between adjacent states. In a slow fading environment, the variations in the received SNR during a symbol period are slow enough that we can consider only adjacent state transitions without incurring a significant penalty. Let $t_{i,j}$ denote the transition probability between states s_i and s_j. Following [3], these can be approximated as:

$$t_{k,k+1} \simeq N_{k+1}/R_s^{(k)}, \quad k = 0, 1, 2, \dots, K-2 \tag{3}$$

$$t_{k,k-1} \simeq N_k/R_s^{(k)}, \quad k = 1, 2, \dots, K-1 \tag{4}$$

where N_k is the theoretical level-crossing rate (LCR) evaluated at Γ_k, and $R_s^{(k)} = \pi_k R_s$ is the average number of symbols transmitted per second during which the SNR is in state s_k, for a symbol rate R_s. We deduce the remaining probabilities using:

$$t_{0,0} = 1 - t_{0,1}, \;\; t_{K-1,K-1} = 1 - t_{K-1,K-2}$$
$$t_{k,k} = 1 - t_{k,k-1} - t_{k,k+1}, \;\; k = 1, 2, \dots, K-2$$

The error probability for each state is calculated as:

$$e_k = \frac{1}{\pi_k} \int_{\Gamma_k}^{\Gamma_{k+1}} e(\alpha) p_\Gamma(\alpha) d\alpha \tag{5}$$

where $e(\gamma)$ is the average error probability for a nonquantized model, conditioned on the SNR. For coherent detection and binary phase-shift keying (CBPSK), $e(\gamma) = Q(\sqrt{2\gamma})$, where $Q(x) = 1/\sqrt{2\pi} \int_x^\infty \exp(-\alpha^2/2) d\alpha$. By changing the order of integration, we can express (5) for the CBPSK case as:

$$e_k = \frac{1}{\pi_k}(\xi_{k+1} - \xi_k) \tag{6}$$

where:

$$\xi_k = F_\Gamma(\Gamma_k) Q(\sqrt{2\Gamma_k}) + I_k \tag{7}$$

$$I_k = I(\Gamma_k) = \int_0^{\Gamma_k} F_\Gamma(\frac{\alpha^2}{2}) \frac{\exp(-\frac{\alpha^2}{2})}{\sqrt{2\pi}} d\alpha \tag{8}$$

Below, we provide analytical expressions for the parameters $\{N_k\}$ and $\{I_k\}$, used in (3)-(4) and (6)-(7) respectively. Once the cdf of the SNR is known, the $\{\Gamma_k\}$ are solved numerically using (1).

B. No Diversity

The pdf and cdf of the received signal SNR for a Nakagami fading channel and no diversity are given by:

$$p_\Gamma(\gamma) = \left(\frac{m}{\bar{\gamma}}\right)^m \frac{\gamma^{m-1}}{\Gamma(m)} e^{-\frac{m}{\bar{\gamma}}\gamma} \tag{9}$$

$$F_\Gamma(\gamma) = \frac{\gamma(m, \frac{m}{\bar{\gamma}}\gamma)}{\Gamma(m)} \tag{10}$$

where m is the fading figure [9], $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function, and $\gamma(x, \alpha) = \int_0^\alpha e^{-t} t^{x-1} dt$ is the incomplete gamma function of the first kind. From [10] we obtain the LCR's:

$$N_k = \sqrt{2\pi} f_m \frac{m^{m-1/2}}{\Gamma(m)} \left(\frac{\Gamma_k}{\bar{\gamma}}\right)^{m-1/2} e^{-m\frac{\Gamma_k}{\bar{\gamma}}} \tag{11}$$

$f_m = v/\lambda_c$ is the maximum Doppler frequency for a vehicle speed v and wavelength λ_c.

Using (8) and (10), we find the following infinite series expression for I_k:

$$I_k = \frac{1}{2\sqrt{\pi}\Gamma(m)} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{m}{\bar{\gamma}}\right)^{m+n}}{n!(m+n)} \gamma(m+n+\frac{1}{2}, \Gamma_k) \tag{12}$$

C. Selection Combining (SC) Diversity

In the rest of the paper we consider only the case of independent diversity channels and identical fading parameters for every channel. The pdf and cdf of the output SNR of a L-branch diversity combiner are:

$$p_\Gamma(\gamma) = L \left[\frac{\gamma(m, \frac{m}{\bar{\gamma}}\gamma)}{\Gamma(m)}\right]^{L-1} \left(\frac{m}{\bar{\gamma}}\right)^m \frac{\gamma^{m-1}}{\Gamma(m)} e^{-\frac{m}{\bar{\gamma}}\gamma} \tag{13}$$

$$F_\Gamma(\gamma) = \left[\frac{\gamma(m, \frac{m}{\bar{\gamma}}\gamma)}{\Gamma(m)}\right]^L \tag{14}$$

The LCR's are obtained as [11]:

$$N_k = \frac{L\sqrt{2\pi} f_m}{[\Gamma(m)]^L} \left(\frac{m}{\bar{\gamma}}\Gamma_k\right)^{m-\frac{1}{2}} e^{-\frac{m}{\bar{\gamma}}\Gamma_k} \left[\gamma(m, \frac{m}{\bar{\gamma}}\Gamma_k)\right]^{L-1} \tag{15}$$

For m integer, using eq. 8.352.1 of [12] and the binomial and multinomial expansions, the cdf can be evaluated as:

$$F_\Gamma(\gamma) = \sum_{l=0}^{L} (-1)^l \binom{L}{l} e^{-l\frac{m}{\bar{\gamma}}\Gamma_k} \sum_{k=0}^{l(m-1)} \beta_{kl} \left(\frac{m}{\bar{\gamma}}\Gamma_k\right)^k \tag{16}$$

where β_{kl} are the coefficients of the multinomial expansion. Using (16) in (8) we obtain:

$$I_k = \frac{1}{2\sqrt{\pi}}\sum_{l=0}^{L}(-1)^l \binom{L}{l} \sum_{k=0}^{l(m-1)} \beta_{kl}\left(\frac{m}{\bar{\gamma}}\right)^k \times \left(\frac{\bar{\gamma}}{lm+\bar{\gamma}}\right)^{k+\frac{1}{2}} \gamma(k+\frac{1}{2}, \frac{lm+\bar{\gamma}}{\bar{\gamma}}\Gamma_k) \quad (17)$$

Recall that this expression is valid only for m integer. For m arbitrary, there is no simple closed-form solution obtainable for I_k, and the e_k's must be calculated numerically via direct integration.

D. Maximal Ratio Combining (MRC) Diversity

With the previous assumption of identical fading parameters, the pdf and cdf of the output SNR are [9]:

$$p_\Gamma(\gamma) = \left(\frac{m_T}{\bar{\gamma_T}}\right)^m \frac{\gamma^{m_T-1}}{\Gamma(m_T)} e^{-\frac{m_T}{\bar{\gamma_T}}\gamma} \quad (18)$$

$$F_\Gamma(\gamma) = \frac{\gamma(m_T, \frac{m_T}{\bar{\gamma_T}}\gamma)}{\Gamma(m_T)} \quad (19)$$

with $m_T = Lm$ and $\bar{\gamma_T} = L\bar{\gamma}$. The desired quantities can be deduced from eqns. (9)-(12) by substituting m with m_T and $\bar{\gamma}$ with $\bar{\gamma_T}$.

E. Equal Gain Combining (EGC) Diversity

We consider only the case of dual-branch diversity ($L = 2$), for which we found [11] the following closed-form and infinite series representations for the pdf and cdf, respectively:

$$p_\Gamma(\gamma) = \frac{B(2m, \frac{1}{2})}{2^{2(m-1)}[\Gamma(m)]^2}\left(\frac{m}{\bar{\gamma}}\right)^{2m} \gamma^{2m-1} e^{-2\frac{m}{\bar{\gamma}}\gamma} \times \Phi(2m, 2m+\frac{1}{2}, \frac{m}{\bar{\gamma}}\gamma) \quad (20)$$

$$F_\Gamma(\gamma) = \frac{\sqrt{\pi}}{[\Gamma(m)]^2 2^{4m-2}} \sum_{n=0}^{\infty} \frac{\Gamma(2m+n)}{\Gamma(2m+n+\frac{1}{2})} \frac{1}{2^n n!} \times \gamma(2m+n, 2\frac{m}{\bar{\gamma}}\gamma) \quad (21)$$

$B(x, y)$ is the beta function and $\Phi(a, c, x)$ the confluent hypergeometric function, given by eqns. 8.380.1 and 9.210.1 of [12], respectively.

We obtained the LCR's as [11]:

$$N_k = \sqrt{2\pi} f_m \frac{B(2m, \frac{1}{2})}{2^{2(m-1)}[\Gamma(m)]^2}\left(\frac{m}{\bar{\gamma}}\Gamma_k\right)^{2m-\frac{1}{2}} e^{-2\frac{m}{\bar{\gamma}}\Gamma_k} \times \Phi(2m, 2m+\frac{1}{2}, \frac{m}{\bar{\gamma}}\Gamma_k) \quad (22)$$

Substituting (21) in (8) results in:

$$I_k = \frac{1}{[\Gamma(m)]^2 2^{4m-1}} \sum_{n=0}^{\infty} \frac{\Gamma(2m+n)}{\Gamma(2m+n+\frac{1}{2})} \frac{1}{2^n n!} \times \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\left(2\frac{m}{\bar{\gamma}}\right)^{2m+n+j}}{2m+n+j} \gamma(2m+n+j+\frac{1}{2}, \Gamma_k)$$

The previous doubly infinite series can become unstable when calculated using limited precision software, thus one might use direct numerical integration to evaluate I_k.

III. Numerical Results

A. Bit Error Rate (BER)

For each diversity type, we compare the BER's obtained via our simulator to the theoretical values for different values of the m parameter. We use $L = 2$ branches for our results, however any number of branches can be accomodated (except for the EGC case). The FSMC has $K = 16$ states. The estimated BER was averaged over 100 simulation runs, each one producing 10^6 samples. Expressions for the error probabilities can be found in [13]. From Figures (1) to (3), we see that the BER's obtained match very well the analytical curves.

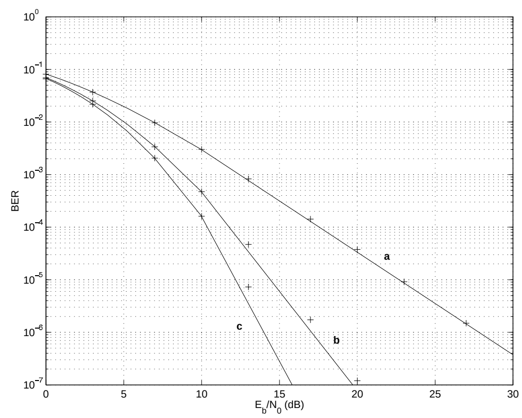

Fig. 1. BER for SC: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

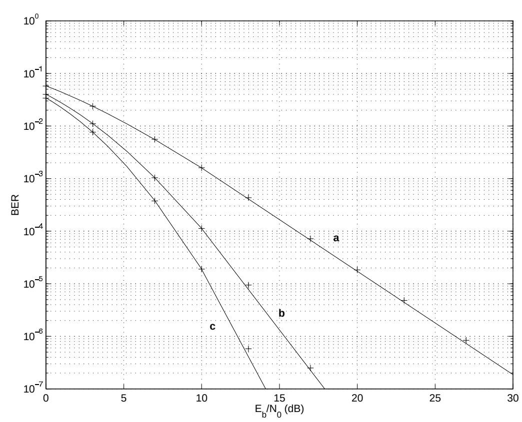

Fig. 2. BER for MRC: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

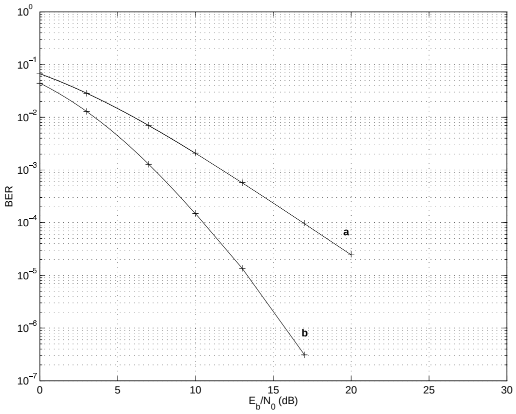

Fig. 3. BER for EGC: (a) $m = 1$, (b) $m = 2$.

B. Level Crossing Rates

In Figures (4)-(6), we compare the LCR's of the FSMC with the theoretical expressions. The LCR's are normalized by f_m, and are plotted as a function of the normalized received envelope $r_n = r/\sqrt{\Omega}$ (in dB). For the FSMC, the value of the channel envelope when the model is in state s_k is computed as $r_k = \sqrt{\bar{\gamma}_k/\bar{\gamma}}$. We used 64 states in order to get sufficient data points, and $L = 2$ diversity branches. As we can see from the plots, the model generates level-crossing statistics very close to the theoretical ones, which is to be expected since the FSMC dynamics are based on the LCR's.

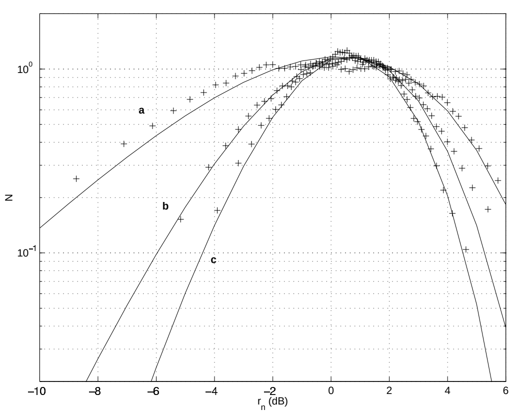

Fig. 4. LCR for SC: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

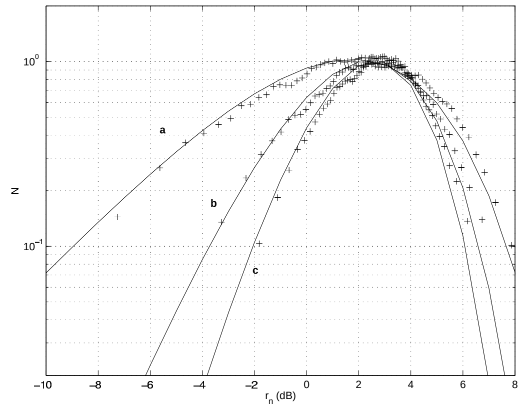

Fig. 5. LCR for MRC: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

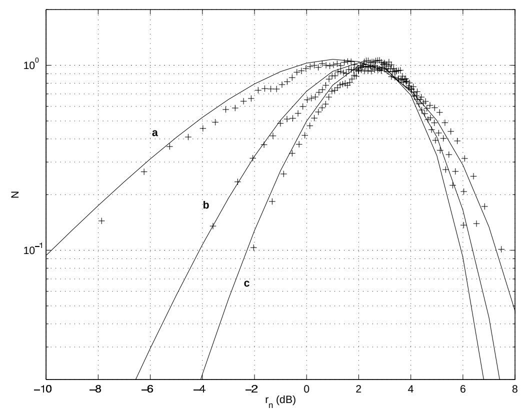

Fig. 6. LCR for EGC: (a) $m = 1$, (b) $m = 2$, (c) $m = 3$.

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