A Model for Stochastic Power Control under Log-normal Distribution

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Abstract — This purpose of this paper is to study the signal-to-interference (SIR) model used in power control problems to incorporate random network matrices with log-normally distributed coefficients. Related models on stochastic programming are presented and applied to the random SIR model.

1 Introduction

The signal-to-interference (SIR) model defined in most of the previous literature on power control in wireless networks, is inherently deterministic. However, in a time-varying channel the channel coefficients are random, leading to a particular form of random SIR model. In these models the network matrices or the desired output (the right hand side of the linear programming model) can be random. Furthermore, the dimension of the network matrices depends on the number users accessing the system, which can also be considered as a random variable. Moreover, even in a stationary channel with deterministic sources or demands there can be estimation or system identification errors which lead to random system model. The stochastic programming approach applied here to the power control problem incorporates some of these random aspects when solving the optimization problem.

The model for stochastic power control in [1] assumed that the interference is Normally distributed. This simplifies the analysis and the power control solution (as the link gains are assumed to be deterministic). However, in general it is natural to consider other distributions. As an example, log-normal distribution has non-negative support and is often used to model long-term fading in wireless networks. This paper presents a solution to optimal power control assuming lognormally distributed network coefficients. The solution is based on a stochastic programming approach to optimal power control.

2 Deterministic Power Control Problem

The purpose of this section is to introduce the notations and the deterministic power control problem, generalized in the next section to the stochastic case. Consider the deterministic power control problem for the uplink in a wireless network [2] with a given number of users m assigned to n base stations. Let $\mathbf{x}$ denote the mn vector of transmit powers. Let $\mathbf{G}$ denote the $m \times mn$ link gain matrix, where m is the number of users in the system and n is the number of receivers. The interference matrix [2] is denoted by $\mathbf{F}$.

In a deterministic uplink power control problem the signal-to-interference ratio (SIR) of each link should exceed or meet the target $\alpha_i, i = 1, ...m$

$$SIR_i = \frac{g_{ii}x_i}{\sum_{j\neq i} r_{ij}g_{ij}x_j + e_i} \geq \alpha_i,\ i = 1, ..., m \quad (1)$$

where m is the number of users in the model, g_{ij} is the non-negative link gain between transmitter j and receiver i, $0 \leq r_{ij} \leq 1$ is a correlation coefficient between user i's and user j's signal waveforms, x_i is the power of transmitter i, and e_i is the external noise power at receiver i. It is assumed here that there is one receiver for each transmitter, but the receivers may or may not be located in the same base station.

It is easy to see [3] that these equations are captured by

$$(\mathbf{\Gamma}^{-1}\mathbf{G} - \mathbf{F})\mathbf{x} \geq \mathbf{e}, \quad (2)$$

where

$$\mathbf{\Gamma} = \mathrm{diag}(\alpha_1, ..., \alpha_m),$$
$$\mathbf{e} = (e_1, ..., e_m)^T,$$
$$\mathbf{G} = \mathrm{diag}(g_{11}, ..., g_{mm})$$

and

$$[\mathbf{F}]_{ij} = \begin{cases} 0 & \text{if } i = j \\ g_{ij}r_{ij} & \text{if } i \neq j \end{cases}$$

in uplink. The configuration of users is feasible if

$$\mathbf{I} - \mathbf{\Gamma}\mathbf{G}^{-1}\mathbf{F} \quad (3)$$

is an M-matrix[4], in which case the solution is given by

$$\mathbf{x} = (\mathbf{I} - \mathbf{\Gamma}\mathbf{G}^{-1}\mathbf{F})^{-1}\mathbf{u}, \quad (4)$$

where $\mathbf{u} = \mathbf{\Gamma}\mathbf{G}^{-1}\mathbf{e}$.

The quality of service (QoS) is measured by the signal-to-noise ratio β. One formulation of the deterministic power control problem can be written as

$$\max_{\mathbf{x}\geq \mathbf{0}} \beta\ s.t.\ \mathbf{G}\mathbf{x} \geq \beta\mathbf{F}\mathbf{x} \quad (5)$$

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such that

$$\sum_{i=1}^{m} x_i \leq R, \tag{6}$$

where R is the given resource constraint.

3 Stochastic Power Control Problem

The purpose of this section is to generalize (5) to allow for uncertainty in the elements of $\mathbf{G}$ and $\mathbf{F}$. In practice the fractional programming problem (5) can be solved by iteratively solving simpler linear programming problems: first, minimizing $\sum_{i=1}^{m} x_i$ for given β such that the QoS inequalities in (5) are satisfied for all rows (users); second, if the solution is feasible ($\sum_{i=1}^{m} x_i \leq R$) β can be increased leading to a new iteration for minimum sum of powers for given QoS restrictions. This iterative scheme is applied in what follows for solving the stochastic version of the original fractional problem (5).

Consider the stochastic resource allocation problem for a communication system (recently discussed in [1]). Write this problem in the general form:

$$\max \beta \tag{7}$$

$$\text{subject to}$$

$$P(\textstyle\sum_i^n g_i x_i - \beta \sum_{i=n+1}^{k} g_i x_i \geq 0) \geq p \tag{8}$$

$$\sum x_i \leq R,$$

$$x_1, ..., x_k \geq 0,$$

$$\beta \geq 0.$$

Assume that $g_1, ..., g_n$ are constants and $\ln(g_{n+1}), .., \ln(g_k)$ are random variables that have a joint normal distribution. The problem (7) is analogous to the problem in [5]; here the randomness concerns the random interference term $I = \sum_{i=n+1}^{k} g_i x_i$ that is multiplied with β whereas in [5] only the vector $(g_1, ..., g_n)$ in (8) is random.

Let $\beta = e^{\gamma}$. Then the problem can be written in the following equivalent form:

$$\max \gamma \tag{9}$$

$$\text{subject to}$$

$$P(\textstyle\sum_{i=1}^{n} g_i x_i - \sum_{i=n+1}^{k} e^{\gamma + h_i + y_i} \geq 0) \geq p$$

$$\textstyle\sum_{i=1}^{n} x_i + \sum_{i=n+1}^{k} e^{y_i} \leq R \tag{10}$$

$$x_1, ..., x_k \geq 0,$$

where $e^{y_i} = x_i, i = n+1, ..., k$, $e^{h_i} = g_i, i = n+1, ..., k$. Note that in problem (9) the variables $\gamma, _{n+1}, ..., y_k$ are unconstrained, e.g., they are not necessarily non-negative. Problems (7) and (9) are equivalent. Theoretically there is some minor mathematics to do here, since if an $x_i, i = n+1, ..., k$ is zero, it cannot be represented in the form $x_i = e^{y_i}$. However, in the application context we can assume that each $x_i, i = n+1, ..., k$ is positive. This is so since a representative user is being considered, and for no user $x_i = 0$ can solve the feasibility part of the capacity maximization problem for $p > 0$.

Theorem 1. *Problem (9) is a convex programming problem.*

Proof. If we consider, for a moment, $h_{n+1}, ..h_k$ as deterministic variables, then

$$\sum_{i=1}^{n} g_i x_i - \sum_{i=n+1}^{k} e^{\gamma + h_i + y_i} \tag{11}$$

is a concave function. In fact, the first part is linear and the second part (without the minus sign) is convex in its argument, hence (9) is convex programming problem by a theorem (see [6]). □

The solution of problem (9) can use the $SUMT$ method [6]. Since this is not likely to be easy (since the appropriate function values can be difficult to obtain) the following section considers an alternative method-of-moments approach to problem (9).

4 Example for Link Coefficients with Lognormal distribution

The more standard assumption on the link coefficients is to assume the coefficients to have a Lognormal distribution. This section presents a straightforward way of transforming a resource allocation problem under the assumption of lognormally distributed coefficients to one with standard normally distributed coefficients analogously to [7].

Let $x_i = e^{y_i} > 0$, $g_i = e^{h_i}$ and assume $\sum_{i=1}^{m} x_i \leq R$. Assume the link coefficients h_is have a joint normal distribution. Consider the problem defined in terms of lognormally distributed coefficients

$$\max \beta \tag{12}$$

$$\text{subject to}$$

$$P(e^{y_i h_i} - \beta \textstyle\sum_{j \neq i} e^{y_j h_j} \geq 0) \geq \bar{p}, \ i = 1, .., m. \tag{13}$$

Problem (12) can be approximated by a model written in terms of normally distributed random variables:

$$\max \beta \tag{14}$$

$$\text{subject to}$$

$$P(\ln(S_i) - \ln(I_i) \geq \ln(\beta)) \geq \bar{p} \ \forall i, \tag{15}$$

where the random variable $\ln(S_i) = y_i h_i$ is normally distributed and $\ln(I_i) = \ln(\sum_{j \neq i} e^{y_j h_j})$ is approximated by a normally distributed random variable using e.g. Wilkinson's moment-matching approach, described in detail in [8] and summarized below. As argued in [9] problem (12) can be rewritten as

$$\min(L(\mathbf{y})) - \phi^{-1}(1 - \bar{p}))\sqrt{C(\mathbf{y})} \tag{16}$$

for given β, where

$$L_i(\mathbf{y}) = \frac{-(m_{S_i} - m_{I_i}) + \ln(\beta)}{\sqrt{C(\mathbf{y})}} \ \forall i; \tag{17}$$

Let $\sqrt{C(\mathbf{y})}$ denote the standard deviation similarly as above and let m_{i1} denote the first moment for the ith random variable (here obtained for the sum of log-normal random variables using Wilkinson's method, as described in [10]). In this approach we replace each probabilistic constraint in problem (12), with given β and R, by a simpler constraint, which is derived by approximating the sum of Lognormal random variables by with a single Lognormal random variable with the same first and second moment. Then, after appropriate logarithmic transformations the problem reduces to one with normal coefficients, as described above. The solution to the *minimum sum of powers* problem is then obtained as the solution to the following nonlinear programming problem:

$$\min \sum x_i \tag{18}$$

subject to

$$-(m_{S_1} - m_{I_1}) + \ln(\beta) - \Phi^{-1}(\bar{q})\sqrt{C_1(\mathbf{y})} \leq 0 \tag{19}$$

$$\vdots \tag{20}$$

$$-(m_{S_m} - m_{I_m}) + \ln(\beta) - \Phi^{-1}(\bar{q})\sqrt{C_m(\mathbf{y})} \leq 0 \tag{21}$$

where for $i = 1, ...m$

$$C_i(\mathbf{y}) = Var(S_i) + Var(I_i)$$

with

$$m_{S_i} = 2\ln(M_{S_{i,1}}) - \frac{1}{2}\ln(M_{S_{i,2}})$$

$$Var(S_i) = \sqrt{\ln(M_{S_{i,2}}) - 2\ln(M_{S_{i,1}})}$$

$$m_{I_i} = 2\ln(M_{I_{i,1}}) - \frac{1}{2}\ln(M_{I_{i,2}})$$

$$Var(I_i) = \sqrt{\ln(M_{I_{i,2}}) - 2\ln(M_{I_{i,1}})}$$

$$M_{S_{i,1}} = e^{\bar{h}_i + y_i + \frac{\sigma_i^2}{2}},$$

$$M_{S_{i,2}} = e^{2\bar{h}_i + y_i + 2\sigma_i^2},$$

$$M_{I_{i,1}} = \sum_{j \neq i} e^{\bar{h}_j + y_j + \frac{\sigma_j^2}{2}},$$

$$\begin{aligned} M_{I_{i,2}} &= \sum_{j \neq i} e^{2\bar{h}_j + y_j + 2\sigma_j^2} \\ &\quad + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} e^{\bar{h}_i + \bar{h}_j} e^{1/2(\sigma_i^2 + \sigma_j^2 + 2r_{ij}\sigma_i^2)}, \end{aligned} \tag{23}$$

where $\bar{q} = 1 - \bar{p}$.

Figure 1 presents numerical results for a case with five users with joint log-normal distribution; the minimum sum of powers is shown to meet the given QoS requirement as given by β and the outage probability p. In the case depicted in the figure each users is in a symmetric channel, with $\bar{h}_i = 0, i = 1, ..., 5$ and $\sigma_i^2 = 1, i = 1, ..., 5$ and coefficients are uncorrelated. In addition external noise power was set to 1. It can be seen that the required transmit power increases rapidly with increasing quality of service parameters. When $\beta = 0.1$ and $\beta = 0.075$ the problem was infeasible when $p \leq 0.45$ and $p \leq 0.4$, respectively.

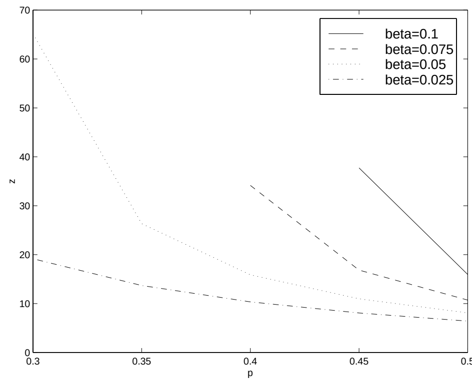

Figure 1: Sum of transmit powers (z) as a function of the outage probability p and β (beta) in a symmetric case when $m = 5$.

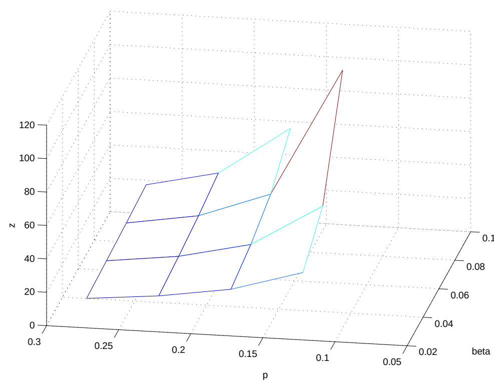

Figure 2: Sum of transmit powers (z) as a function of the outage probability p with different values of β (beta) in a symmetric one dimensional network when $m = 5$

Figure 2 presents numerical results for a one dimensional network with five users and joint log-normal network coefficients. The minimum sum of powers required to meet the given QoS, as given by $0.025 \leq \beta \leq 0.1$ and $0.125 \leq p \leq 0.275$, is depicted. In this experiment each users is in a symmetric channel with $g_{ij} = d_{ij}^{-4} e^{h_{ij}}$, $h_{ij} \sim N(\bar{h}_{ij}, \sigma_{ij}^2)$. The mean is defined by

$$E(g_{ij}) = d_{ij}^{-4} \exp(\bar{h}_{ij}) \tag{24}$$

with $\bar{h}_{ij} = 0, i = 1, ..., 5$ $j = 1, ..., 5$, and with matrix

$\mathbf{D} = (d_{ij})$

$$\mathbf{D} = \begin{bmatrix} 1 & 3 & 5 & 3 & 1 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 5 \\ 5 & 3 & 1 & 1 & 3 \\ 3 & 5 & 3 & 1 & 1 \end{bmatrix}.$$

The elements of matrix $\mathbf{D}$ define the distances between each transmitter (MS) and each receiver (BS). Here it is assumed that the distance between the base stations is two kilometers and the distance between each transmitter and the nearest receiver is one kilometer. The propagation loss model $1/d^4$, adopted e.g. in [2], is used in the definition of the mean in equation (24). That is, the interference and signal power received by a base station at distance d from a given transmitter is reduced by factor $1/d^4$.

The covariance term is given by

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Thus, all coefficients are uncorrelated. Here the interference inflicted on the two most distant base stations is calculated only once, which is a convenient approximation and causes negligible effect to the results. In addition external noise power was set to 1. It can be seen that the spatial separation between the users and receivers enables the system to achieve much higher quality of service in terms of p and β than for example in the case shown in Figure 1, where all users are received at the same base station. In Figure 2 only the case when $(\beta, p) = (0.1, 0.075)$ in infeasible and subsequently omitted from the figure. In this case the used nonlinear programming algorithm increased transmit powers for each user until the maximum allowed power was reached.

The aforementioned results were obtained using Wilkinson's approximation in evaluating the probabilistic constraints. This approximation may not be appropriate or most convenient in all cases. In particular, when the number of users m in the system is large, the central limit theorem suggests solving

$$\min \sum x_i \tag{25}$$

subject to

$$-(m_{S_1} - m_{I_1}\beta) - \Phi^{-1}(1-\bar{p})\sqrt{\mathbf{C}_1(\mathbf{y})} \leq 0 \tag{26}$$

$$\vdots \tag{27}$$

$$-(m_{S_m} - m_{I_m}\beta) - \Phi^{-1}(1-\bar{p})\sqrt{\mathbf{C}_m(\mathbf{y})} \leq 0 \tag{28}$$

where

$$m_{S_i} = e^{\bar{h}_i + y_i + \frac{\sigma_i^2}{2}},\ i = 1, ..., m,$$

$$m_{I_i} = \sum_{j \neq i} e^{\bar{h}_j + y_j + \frac{\sigma_j^2}{2}},\ i = 1, ..., m,$$

and

$$\mathbf{C}_i = e^{\sigma_i^2} + \beta^2(M_{I_{i,2}} - (M_{I_{i,1}})^2).$$

The application of the normal approximation is left for further study. It is unlikely the results are improved by this approach, but in some cases the solutions can potentially be further simplified.

5 Conclusion

This paper has introduced a model for optimum stochastic power control under lognormally distributed link gain coefficients. Adding a covariance matrix to the analysis leads to a solution requiring relatively more transmit energy to satisfy the QoS restrictions, as compared to a traditional solution to stochastic power control.

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